On the dynamics of unsteady gravity waves of finite amplitude

Part 2. Local properties of a random wave field

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Expressions in closed form are derived for a number of local properties of a random, irrotational wave field. They are: (i) the mean potential and kinetic energies per unit projected area; (ii) the energy balance among the processes of energy input from the surface pressure fluctuations, rate of growth of potential and kinetic energy and horizontal energy flux; and (iii) the partition between potential and kinetic energy. These expressions are mainly in terms of quantities measured at the free surface, which are therefore functions of only two spatial variables (x, y) and of time t.

Approximations for these expressions can be found simply by subsequent expansion methods; the fourth order being the highest for which the assumption of irrotational motion is appropriate in a real fluid. It is shown that the mean product of any three first-order quantities is of fourth or higher order in the root-mean-square wave slope, and this result is applied in estimating the magnitude of some higher order effects. In particular, the skewness of the surface displacement is of the order of the root-mean-square surface slope, which has been confirmed observationally by Kinsman (1960).

1. Introduction

This paper is concerned with the specification of properties of a random wave field of finite amplitude that are associated with a single point in the horizontal plane, and with the interrelations among them. The problem of the dynamical interactions among the various wave components is a complex one; the object of this paper is to derive a number of simple basic expressions and results from which a more detailed analysis can be developed. The usual method of approach to such problems involves *ab initio* an expansion about the equilibrium free surface and an investigation of successively higher order terms. This was the technique used in Part 1 of this sequence (Phillips 1960*a*), concerned with the elementary wave interactions, where it was shown that continuing energy transfer from one Fourier component of the wave field to another was possible through a certain type of resonant third-order interaction. As the order of the approximation increases, however, so does the algebraic complexity—very rapidly indeed—so that it is very desirable to carry the analysis in closed form as far as

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possible. This is accomplished in the present paper by expressing properties of the three-dimensional wave motion in terms of quantities measured on the free surface at a fixed horizontal location, which are functions of only two spatial variables (x, y) and of time. Typical quantities of this kind are the surface displacement $\xi(x, y, t)$ itself, and, say, the velocity potential at the free surface

$$[\phi(x, y, z, t)]_{z=\xi(x, y, t)}.$$

Clearly, the specification of quantities in this way differs from both an Eulerian and a Lagrangian description, since the horizontal co-ordinates are fixed but the vertical co-ordinate moves up and down with the free surface.

A number of expressions of a type somewhat similar to those derived in this paper have been given by Starr (1947). He considered periodic or solitary waves and was concerned with finding relations among the wave energy, phase velocity, wave momentum, etc. These relations, however, cannot be generalized to the case of a random wave field, nor are Starr's methods capable of treating the multi-point properties that occur in the full dynamical problem.

The technique to be developed here results in higher order effects (i.e. those that do not appear in the infinitesimal wave theory) presenting themselves frequently as mean products of three or more first-order quantities. To obtain a first approximation for these higher-order effects, only the infinitesimal wave solution is required, and a considerable gain in simplicity is achieved. An elementary application of this technique was made in a recent paper (Phillips 1960*b*) where the drift velocity and mean surface velocity in an irrotational wave field were found very readily. In cases where these effects are represented by the mean product of three first-order terms, the theorem derived in § 5 shows that they are quite generally of the fourth order, not the third as might at first appear. This theorem is used to estimate the orders of magnitude of the skewness of the surface displacement and of the difference between the kinetic and potential energies per unit area of the wave field.

Although the results given by this approach are in closed form and 'exact' within the context of potential theory, it is well known (Longuet-Higgins 1953) that in a real fluid, the motion is not truly irrotational. A mean vorticity field, of second order, diffuses inwards from the free surface (and also perhaps from the bottom), ultimately affecting the whole fluid. This results in, for example, a mass transport velocity in a two-dimensional motion that is different from the value found by Stokes for potential motion, even when the fluid viscosity is vanishingly small. Also, in oceanographical contexts where the water motion is three-dimensional, one must expect the vorticity field to become distorted or convoluted, resulting in a turbulent motion that is maintained by the rate of strain associated with the wave field. This turbulent layer near the surface has been shown (Phillips 1961) to influence the momentum equation for potential motion at the third order, and results in a slow attenuation of the waves. In the energy equation (discussed in some detail in § 3), which is formed by multiplying the momentum equation by the first-order surface velocity $\dot{\xi}$, the influence is thus of the fourth order. As a consequence, the exactness implied by the closed form of the expressions given below is partly illusory, and the energy relations cannot be expected to be physically reliable beyond the fourth order. However, their concise form makes them convenient for subsequent approximation to this order.

2. The basic equations for finite amplitude irrotational waves

Consider a field of gravity waves on the surface of an incompressible liquid in which the depth is large compared with any wavelengths involved.† It will be supposed that the motion is irrotational, keeping in mind the conclusions of the previous section. A Cartesian reference frame will be used, with the z-axis vertically upwards and the plane $z = 0$ corresponding to the equilibrium free surface. Let $\mathbf{q}$ represent the velocity of the fluid at an interior point $\mathbf{x} = (x, y, z)$ so that

$$\left.\begin{aligned} \mathbf{q} &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \\ \text{and} \qquad \frac{\partial^2\phi}{\partial x^2} &+ \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \end{aligned}\right\} \tag{2.1}$$

The pressure at any point in the fluid is given by Bernoulli's equation

$$\frac{p}{\rho} + \frac{\partial\phi}{\partial t} + \tfrac{1}{2}q^2 + gz = 0. \tag{2.2}$$

If surface tension is neglected, the pressure is continuous across the free surface $z = \xi(x, y, t)$, so that from equation (2.2) we obtain the dynamical surface boundary condition

$$-\frac{p}{\rho} = \left(\frac{\partial\phi}{\partial t}\right)_\xi + \tfrac{1}{2}q_\xi^2 + g\xi, \tag{2.3}$$

where $p(x, y, t)$ now represents the departure of the atmospheric surface pressure from its mean value, ρ the water density and the subscript ξ indicates that a quantity is to be measured at the free surface. There is, in addition, a kinematical boundary condition

$$\begin{aligned} \left(\frac{\partial\phi}{\partial z}\right)_\xi &= \frac{D\xi}{Dt} \\ &= \frac{\partial\xi}{\partial t} + (\nabla\phi)_\xi . \nabla\xi, \end{aligned} \tag{2.4}$$

where, here and for the remainder of this paper the two-dimensional operator $\nabla \equiv (\partial/\partial x, \partial/\partial y)$. If $\mathbf{u} = \nabla\phi$, the horizontal (vectorial) component of the fluid velocity $\mathbf{q}$ and $w = \partial\phi/\partial z$, the vertical component of $\mathbf{q}$, equation (2.4) can be expressed as

$$w_\xi = \dot{\xi} + \mathbf{u}_\xi . \nabla\xi. \tag{2.5}$$

Two further differential relations will be used frequently in the following analysis. If

$$f_\xi \equiv [f(x, y, z, t)]_{z=\xi(x,y,t)} \tag{2.6}$$

† Cases of finite (even variable) depth can be considered formally in a manner similar to that given here, but the influence of the vorticity field may be more serious.

represents some function measured at the free surface, then by the rules of partial differentiation,

$$\left.\begin{aligned}\frac{\partial}{\partial t}(f_\xi) &= \left(\frac{\partial f}{\partial t}\right)_\xi + \left(\frac{\partial f}{\partial z}\right)_\xi \frac{\partial \xi}{\partial t},\\ \nabla(f_\xi) &= (\nabla f)_\xi + \left(\frac{\partial f}{\partial z}\right)_\xi \nabla\xi.\end{aligned}\right\} \tag{2.7}$$

Clearly, the operations $(\ \ )_\xi$ and differentiation do not commute; a little care is sometimes necessary as a result. Important special cases of (2.7) are given when $f = \phi$, the velocity potential

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\phi_\xi) &= \dot{\phi}_\xi + w_\xi \dot{\xi},\\ \nabla(\phi_\xi) &= \mathbf{u}_\xi + w_\xi \nabla\xi.\end{aligned}\right\} \tag{2.8}$$

Also, by differentiation of (2.5), we have, with the aid of (2.7),

$$\dot{w}_\xi = \ddot{\xi} + \dot{\mathbf{u}}_\xi . \nabla\xi + \nabla . (\dot{\xi}\mathbf{u}_\xi), \tag{2.9}$$

concerning the acceleration of the free surface.

3. Single point analysis—energy relations

(i) *Potential and kinetic energy densities*

The potential energy per unit horizontal area, taking as a datum the equilibrium plane $z = 0$, is simply

$$\mathscr{V} = \tfrac{1}{2}\rho g \xi^2, \tag{3.1}$$

and the mean potential energy per unit projected area, or the potential energy density, is

$$V = \tfrac{1}{2}\rho g \overline{\xi^2}, \tag{3.2}$$

where the bar denotes a probability or ensemble average. If the wave field is statistically stationary in time (as, for example, when a steady wind blows for a long time over a finite fetch) this average is equal to the usual time average. Similarly, if the wave field is spatially homogeneous, the probability average is equal to the space average. The expression (3.2) is valid quite generally, with the appropriate interpretation of the averaging process.

The kinetic energy of the wave motion per unit projected area is given by

$$\mathscr{T} = \tfrac{1}{2}\rho \int_{-\infty}^{\xi} q^2\, dz. \tag{3.3}$$

From (3.3),

$$\begin{aligned}\frac{2}{\rho}\mathscr{T} &= \int_{-\infty}^{\xi} \left\{(\nabla\phi)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dz,\\ &= \int_{-\infty}^{\xi} \left\{\nabla . (\phi\nabla\phi) - \phi\nabla^2\phi + \frac{\partial}{\partial z}\left(\phi\frac{\partial\phi}{\partial z}\right) - \phi\frac{\partial^2\phi}{\partial z^2}\right\} dz,\\ &= \phi_\xi w_\xi + \int_{-\infty}^{\xi} \nabla . (\phi\nabla\phi)\, dz,\end{aligned}$$

since

$$\nabla^2\phi + \frac{\partial^2\phi}{\partial z^2} = 0.$$

Furthermore, $$\nabla . \int_{-\infty}^{\xi} \phi \nabla \phi \, dz = \nabla \xi . (\nabla \phi)_\xi \phi_\xi + \int_{-\infty}^{\xi} \nabla . (\phi \nabla \phi) \, dz,$$

so that $$\frac{2}{\rho} \mathscr{T} = \phi_\xi \{w_\xi - \mathbf{u}_\xi . \nabla \xi\} + \nabla . \int_{-\infty}^{\xi} \phi \nabla \phi \, dz,$$
$$= \phi_\xi \dot{\xi} + \nabla . \int_{-\infty}^{\xi} \phi \mathbf{u} \, dz, \tag{3.4}$$

in virtue of (2.5). The mean kinetic energy density of the wave field is thus

$$T = \tfrac{1}{2}\rho \overline{\phi_\xi \dot{\xi}} + \tfrac{1}{2}\rho \nabla . \overline{\int_{-\infty}^{\xi} \phi \mathbf{u} \, dz}. \tag{3.5}$$

If the wave field is spatially homogeneous, the mean value of the integral is independent of position, and the previous expression becomes

$$T = \tfrac{1}{2}\rho \overline{\phi_\xi \dot{\xi}}. \tag{3.6}$$

These expressions can be obtained alternatively by using the relation

$$\mathscr{T} = \tfrac{1}{2}\rho \int \phi \frac{\partial \phi}{\partial n} dS,$$

where the integral is over the surface of a vertical prism of unit cross-sectional area terminating at the moving free surface $z = \xi(x, y, t)$.

(ii) *Rate of change of kinetic energy density*

The time rate of change of the mean kinetic energy per unit projected area can be expressed in a number of alternative and equivalent ways. One such form is obtained as follows:

$$\frac{2}{\rho}\frac{\partial \mathscr{T}}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{\xi} q^2 \, dz,$$
$$= \dot{\xi} q_\xi^2 + 2 \int_{-\infty}^{\xi} \mathbf{q} . \dot{\mathbf{q}} \, dz. \tag{3.7}$$

But $$\int_{-\infty}^{\xi} \mathbf{q} . \dot{\mathbf{q}} \, dz = \int_{-\infty}^{\xi} \left\{ \nabla \phi . \nabla \dot{\phi} + \frac{\partial \phi}{\partial z} \frac{\partial \dot{\phi}}{\partial z} \right\} dz$$
$$= \int_{-\infty}^{\xi} \left\{ \nabla . (\dot{\phi} \nabla \phi) - \dot{\phi} \nabla^2 \phi + \frac{\partial}{\partial z}\left(\dot{\phi} \frac{\partial \phi}{\partial z} \right) - \dot{\phi} \frac{\partial^2 \phi}{\partial z^2} \right\} dz$$
$$= \dot{\phi}_\xi w_\xi + \int_{-\infty}^{\xi} \nabla . (\dot{\phi} \nabla \phi) \, dz$$
$$= \dot{\phi}_\xi w_\xi + \nabla . \int_{-\infty}^{\xi} \dot{\phi} \nabla \phi \, dz - \dot{\phi}_\xi \nabla \xi . (\nabla \phi)_\xi$$
$$= \dot{\phi}_\xi \dot{\xi} + \nabla . \int_{-\infty}^{\xi} \dot{\phi} \mathbf{u} \, dz \tag{3.8}$$

from (2.5). Thus $$\dot{\mathscr{T}} = \rho \dot{\phi}_\xi \dot{\xi} + \tfrac{1}{2}\rho q_\xi^2 \dot{\xi} + \nabla . \int_{-\infty}^{\xi} \rho \dot{\phi} \mathbf{u} \, dz. \tag{3.9}$$

If the wave field is spatially homogeneous, the average of equation (3.9) yields

$$\dot{T} = \rho \overline{\dot{\phi}_\xi \dot{\xi}} + \tfrac{1}{2}\rho \overline{q_\xi^2 \dot{\xi}}, \tag{3.10}$$

since the mean of the integral is independent of position (x, y), whereas for a field statistically steady in time, $\dot{T} = 0$ and

$$-\nabla . \overline{\int_{-\infty}^{\xi} \rho\phi \mathbf{u}\, dz} = \rho\, \overline{\phi_\xi \dot{\xi}} + \tfrac{1}{2}\rho\, \overline{q_\xi^2 \dot{\xi}}. \tag{3.11}$$

These particular forms lead to simple physical interpretations that are of some interest. Equation (3.10), say, gives the mean rate of change of the kinetic energy in a column of fluid of unit cross-sectional area as the sum of two terms. The first arises from the rate of change of q^2 in the column and the second from the rate of change of length (and so mass) of the column. These are exactly equivalent to the two terms arising in

$$\frac{\partial}{\partial t}(\tfrac{1}{2}mq^2) = m\mathbf{q}\,.\,\dot{\mathbf{q}} + \tfrac{1}{2}\dot{m}q^2.$$

The integral in equation (3.9) can be shown to represent the negative flux of energy per unit horizontal length, or minus the energy flux across a vertical surface of unit width and infinite depth. To demonstrate this, we must add the rate of working by pressure forces across this surface to the rate of convection of kinetic energy across the surface by the motion. Consider, then, a vertical strip cutting the equilibrium free surface in the line element $d\mathbf{s}$. The difference between the pressure at any point and the undisturbed hydrostatic pressure is $(p+\rho gz)$, so that the rate at which work is done by pressure forces acting across the strip is

$$d\mathbf{s}\,.\int_{-\infty}^{\xi} (p+\rho gz)\,\mathbf{u}\,dz.$$

The rate at which kinetic energy is convected across the strip is

$$d\mathbf{s}\,.\int_{-\infty}^{\xi} \tfrac{1}{2}\rho q^2 \mathbf{u}\,dz.$$

The total flux energy across the strip is thus

$$d\mathbf{s}\,.\int_{-\infty}^{\xi} (p+\tfrac{1}{2}\rho q^2+\rho gz)\,\mathbf{u}\,dz = -d\mathbf{s}\,.\int_{-\infty}^{\xi} \rho\frac{\partial\phi}{\partial t}\mathbf{u}\,dz,$$

from Bernoulli's equation (2.2). The instantaneous energy flux vector is thus

$$-\int_{-\infty}^{\xi} \rho\phi\mathbf{u}\,dz, \tag{3.12}$$

and the mean energy flux is

$$\mathbf{F} = -\overline{\int_{-\infty}^{\xi} \rho\phi\mathbf{u}\,dz}. \tag{3.13}$$

It appears, then, that the left-hand side of equation (3.11) can be interpreted as the horizontal divergence of the mean energy-flux vector $\mathbf{F}$. A more general expression, valid for a non-stationary, non-homogeneous wave field is obtained by taking the probability average of equation (3.9),

$$\dot{T} + \nabla . \mathbf{F} = \rho\, \overline{\phi_\xi \dot{\xi}} + \tfrac{1}{2}\rho\, \overline{q_\xi^2 \dot{\xi}}. \tag{3.14}$$

An alternative expression for $\dot{T}$ can be found by returning to (3.7) and (3.8).

$$\int_{-\infty}^{\xi} \mathbf{q}\cdot\dot{\mathbf{q}}\,dz = \int_{-\infty}^{\xi}\left\{\nabla\cdot(\phi\nabla\dot{\phi})-\phi\nabla^2\dot{\phi}+\frac{\partial}{\partial z}\left(\phi\frac{\partial\dot{\phi}}{\partial z}\right)-\phi\frac{\partial^2\dot{\phi}}{\partial z^2}\right\}dz$$

$$= \phi_\xi\left(\frac{\partial\dot{\phi}}{\partial z}\right)_\xi - \phi_\xi\nabla\xi\cdot(\nabla\dot{\phi})_\xi + \nabla\cdot\int_{-\infty}^{\xi}\phi\nabla\dot{\phi}\,dz$$

$$= \phi_\xi(\dot{w}_\xi - \dot{\mathbf{u}}_\xi\cdot\nabla\xi) + \nabla\cdot\int_{-\infty}^{\xi}\phi\dot{\mathbf{u}}\,dz.$$

The use of (2.9) immediately, and then of (2.8) and (2.5) leads to

$$\int_{-\infty}^{\xi}\mathbf{q}\cdot\dot{\mathbf{q}}\,dz = \phi_\xi\ddot{\xi} - \dot{\xi}q_\xi^2 + w_\xi\dot{\xi}^2 + \nabla\cdot\{\phi_\xi\dot{\xi}\mathbf{u}_\xi\} + \nabla\cdot\int_{-\infty}^{\xi}\phi\dot{\mathbf{u}}\,dz,$$

which, after addition of $\frac{1}{2}\dot{\xi}q_\xi^2$, represents $\rho^{-1}\dot{\mathscr{T}}$, as (3.7) indicates. In a homogeneous wave field, the mean value of this expression simplifies since both of the divergence terms vanish, and we find that

$$\dot{T} = \rho\overline{\phi_\xi\ddot{\xi}} + \rho\overline{(w_\xi\dot{\xi} - \tfrac{1}{2}q_\xi^2)\,\dot{\xi}}. \tag{3.15}$$

Adding this result to the expression (3.10), previously obtained, we have

$$\dot{T} = \tfrac{1}{2}\rho\{\overline{\dot{\phi}_\xi\dot{\xi}} + \overline{\phi_\xi\ddot{\xi}} + \overline{w_\xi\dot{\xi}^2}\}, \tag{3.16}$$

which, as the reader may verify, can be obtained more readily by direct differentiation of (3.6).

(iii) *The energy balance*

If we multiply the dynamic free surface condition (2.3) by $\dot{\xi}$ and take the probability average of the result, we obtain

$$-\overline{p\dot{\xi}} = \rho\overline{\dot{\phi}_\xi\dot{\xi}} + \tfrac{1}{2}\rho\overline{q_\xi^2\dot{\xi}} + \rho g\overline{\xi\dot{\xi}}, \tag{3.17}$$

which can be interpreted as the equation for the mean energy balance for a field of gravity waves without restriction to stationarity in time or homogeneity in space. The term $-\overline{p\dot{\xi}}$ represents the rate at which energy is supplied to the waves per unit projected area by the atmospheric pressure fluctuations. The first and second terms on the right-hand side together represent, from equation (3.14), $\dot{T}+\nabla\cdot\mathbf{F}$, the rate of change of the mean kinetic energy per unit projected area plus the horizontal divergence of the mean energy flux vector. The last term clearly represents $\dot{V}$, the rate of change of mean potential energy per unit projected area. Equation (3.17) can therefore be written

$$-\overline{p\dot{\xi}} = \dot{T} + \dot{V} + \nabla\cdot\mathbf{F}. \tag{3.18}$$

If the wave field is homogeneous, $\mathbf{F}$ is constant,

$$-\overline{p\dot{\xi}} = \dot{T} + \dot{V}, \tag{3.19}$$

and if it is statistically steady

$$-\overline{p\dot{\xi}} = \nabla\cdot\mathbf{F}. \tag{3.20}$$

(iv) *Energy partitions in a homogeneous wave field*

In a homogeneous wave field of infinitesimal amplitude, the total energy per unit projected area is divided equally between potential and kinetic energy. For finite amplitude waves, this is in general not so, though the difference is small. The determination of $T-V$ has certain intrinsic interest [Starr (1947) and Mack (1958) having considered certain special cases], but our object here is simply to devise a method of eliminating the kinetic energy density from our dynamical considerations.

The required expression is obtained by multiplying the dynamical surface boundary condition (2.3) by the surface displacement $\xi(x, y, t)$:

$$\begin{aligned}\rho^{-1}p\xi + \tfrac{1}{2}q_\xi^2\,\xi + g\xi^2 &= -\xi\left(\frac{\partial\phi}{\partial t}\right)_\xi \\ &= -\xi\frac{\partial}{\partial t}(\phi_\xi) + w_\xi\dot{\xi}\xi \\ &= -\frac{\partial}{\partial t}(\xi\phi_\xi) + \dot{\xi}\phi_\xi + w_\xi\dot{\xi}\xi.\end{aligned}$$

Rearranging the terms and averaging the whole equation, we find that, for a homogeneous wave field,

$$V = T + \tfrac{1}{2}\rho\,\overline{\xi(w_\xi\dot{\xi} - \tfrac{1}{2}q_\xi^2)} - \tfrac{1}{2}\overline{p\xi} - \tfrac{1}{2}\rho\frac{\partial}{\partial t}(\overline{\xi\phi_\xi}). \tag{3.21}$$

Under most physical circumstances, the last two terms can be neglected. The term $\frac{1}{2}\overline{p\xi}$ is clearly the contribution to V from the air pressure fluctuations acting hydrostatically. Some recent unpublished observations by Dr M. S. Longuet-Higgins indicate that $(\overline{p^2})^{\frac{1}{2}}$ is of order $\rho_a g(\overline{\xi^2})^{\frac{1}{2}}$ under typical sea conditions, where ρ_a is the air density. The ratio of this term to V is then of order $\rho_a/\rho_w \sim 10^{-3}$, which we can neglect. The ratio of the last term to the kinetic energy term T is of order τ/θ where τ is a characteristic wave period and θ the time scale of growth of the wave field as a whole. Recent observational evidence indicates that this ratio is of order 10^{-3} for the shorter components of a wind-generated wave field (those which travel at a phase speed less than half the speed of the wind) and may be as small as 10^{-5} for the longer components. It appears then that the energy partition can be expressed to sufficient accuracy as

$$V = T + \tfrac{1}{2}\rho\,\overline{\xi(w_\xi\dot{\xi} - \tfrac{1}{2}q_\xi^2)}. \tag{3.22}$$

With this expression, the kinetic energy density can be eliminated from the energy balance in a homogeneous wave field, yielding

$$2\dot{V} - \tfrac{1}{2}\rho\frac{d}{dt}\{\overline{\xi(w_\xi\dot{\xi} - \tfrac{1}{2}q_\xi^2)}\} = -\overline{p\dot{\xi}}. \tag{3.23}$$

This equation gives the rate of growth of the potential energy density (the most accessible physical observable) directly in terms of the rate of energy supply and the small 'unbalance' term.

Another type of energy partition is perhaps worth noting in passing. In a homogeneous wave field, for any fixed point always below the water surface,

$$\overline{\mathbf{u}^2} = \overline{w^2} = \tfrac{1}{2}\overline{q^2}, \tag{3.24}$$

so that the kinetic energy per unit mass at such points is divided equally between the horizontal and vertical motions. This result was proved (Phillips 1955) in a rather different, but equivalent context.

4. The energy-flux vector

As an illustration of the use of these expressions in finding lowest-order approximations, let us relate the energy-flux vector $\mathbf{F}$ to the spectrum of surface displacements (or the potential energy spectrum, which differs only by the factor $\frac{1}{2}\rho g$).

The surface displacement $\xi(\mathbf{x}, t)$ can be represented quite generally by the generalized Fourier transform

$$\xi(\mathbf{x}, t) = \int_{\mathbf{k}}\int_{n} B(\mathbf{k}, n)\, e^{i(\mathbf{k}.\mathbf{x}-nt)}\, d\mathbf{k}\, dn, \tag{4.1}$$

where the integrations range over the entire wave-number plane and over all values of frequency n. A clear and concise account of the modern theory of generalized Fourier transforms is given by Lighthill (1958). In free infinitesimal wave theory, the generalized Fourier transform $B(\mathbf{k}, n)$ can be shown simply to be of the form

$$B(\mathbf{k}, n) = A(\mathbf{k})\, \delta(n-\omega), \tag{4.2}$$

where δ represents the Dirac delta function, $\omega = (gk)^{\frac{1}{2}}$ and we have adopted the convention that the direction of propagation of a wave is in the direction of $\mathbf{k}$. The expression (4.2) is simply a statement of the fact that, in infinitesimal wave theory, a wave-number k has associated with it a unique frequency ω.

To the first order, then, the surface displacement is given from (4.1) and (4.2) by

$$\left.\begin{aligned} \xi(\mathbf{x}, t) &= \int_{\mathbf{k}} A(\mathbf{k})\, e^{i(\mathbf{k}.\mathbf{x}-\omega t)}\, d\mathbf{k} \\ &= \int_{\mathbf{k}} A^*(\mathbf{k})\, e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\, d\mathbf{k}, \end{aligned}\right\} \tag{4.3}$$

since ξ is real, where the asterisk represents the complex conjugate quantity. The associated velocity potential, correct to first order, is given by

$$\phi(\mathbf{x}, z, t) = -i\int \frac{\omega}{k} A(\mathbf{k})\, e^{kz}\, e^{i(\mathbf{k}.\mathbf{x}-\omega t)}\, d\mathbf{k}. \tag{4.4}$$

The relation between the surface displacement spectrum $\Phi(\mathbf{k})$ and the surface displacement covariance

$$\Xi(\mathbf{x}, \mathbf{r}; t, \tau) = \overline{\xi(\mathbf{x}, t)\, \xi(\mathbf{x}+\mathbf{r}, t+\tau)} \tag{4.5}$$

is constructed by multiplying (4.3) by a corresponding expression for $\xi(\mathbf{x}+\mathbf{r}, t+\tau)$. If the wave field approximates sufficiently to stationarity and homogeneity, we have from the theory of generalized functions

$$\overline{A^*(\mathbf{k})\, A(\mathbf{k}')} = \Phi(\mathbf{k})\, \delta(\mathbf{k}-\mathbf{k}'), \tag{4.6}$$

where $\delta(\mathbf{k}-\mathbf{k}')$ represents the product of the Dirac delta functions involving the two components of the vector argument. Thus

$$\Xi(\mathbf{r},\tau) = \int_{\mathbf{k}} \Phi(\mathbf{k})\, e^{i(\mathbf{k}.\mathbf{r}-\omega\tau)}\, d\mathbf{k}. \tag{4.7}$$

In particular, when the time interval $\tau = 0$,

$$\Xi(\mathbf{r},0) = \int_{\mathbf{k}} \Phi(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}\, d\mathbf{k}, \tag{4.8}$$

and when $\mathbf{r} = 0$, we have the frequency spectrum for the surface oscillations measured at a point

$$\Xi(0,\tau) = \int_{\mathbf{k}} \Phi(\mathbf{k})\, e^{-i\omega\tau}\, d\mathbf{k} = \int_0^\infty \int_0^{2\pi} \Phi(\mathbf{k})\, e^{-i\omega\tau}\, k\, dk\, d\theta,$$

where the integration is taken in polar co-ordinates. But, since $\omega^2 = gk$,

$$k\,dk = \frac{2\omega^3 d\omega}{g^2},$$

so that

$$\Xi(0,\tau) = \int_0^\infty \Psi(\omega)\, e^{i\omega\tau}\, d\omega, \tag{4.9}$$

where

$$\Psi(\omega) = \Psi^*(\omega) = \frac{2\omega^3}{g^2} \int_0^{2\pi} \Phi(\mathbf{k})\, d\theta \tag{4.10}$$

and $k = |\mathbf{k}| = \omega^2/g$. The spectrum $\Psi(\omega)$ is measured fairly readily using a conventional wave pole.

In a similar manner, we can construct an expression for the energy-flux vector $\mathbf{F}$ in terms of $\Phi(\mathbf{k})$. From (3.13),

$$\mathbf{F} = -\rho \overline{\int_{-\infty}^{\xi} \phi \mathbf{u}\, dz} = -\rho \int_{-\infty}^{0} \overline{\phi \mathbf{u}}\, dz,$$

correct to second order. From (4.4),

$$\mathbf{F} = \rho \int_{-\infty}^{0} dz \int_{\mathbf{k}} \int_{\mathbf{k}'} \frac{\omega^2}{k} \frac{\omega' \mathbf{k}'}{k'} \overline{A^*(\mathbf{k})\, A(\mathbf{k}')}\, e^{(k+k')z}$$
$$\times \exp i\{(\mathbf{k}'-\mathbf{k}).\mathbf{x} - (\omega'-\omega)t\}\, d\mathbf{k}\, d\mathbf{k}',$$

where $\omega' = (gk')^{\frac{1}{2}}$ so that, in view of (4.6),

$$\mathbf{F} = \rho \int_{\mathbf{k}} \frac{\omega^3 \mathbf{k}}{k^2} \Phi(\mathbf{k}) \left\{ \int_{-\infty}^{0} e^{2kz}\, dz \right\} d\mathbf{k}$$
$$= \tfrac{1}{2}\rho g \int \mathbf{c}(k)\, \Phi(\mathbf{k})\, d\mathbf{k}, \tag{4.11}$$

where $\mathbf{c}(k) = \omega \mathbf{k}/k^2$ is the phase velocity of waves of wave-number $\mathbf{k}$. This is a simple generalization of the statement that, in a simple sinusoidal wave, the energy flux is equal to the potential energy density moving at the phase velocity, or to the total energy moving at half the phase velocity (the group velocity). The expression (4.11) is correct to second order; the next approximation could be found by substituting the next-order solutions for $\phi(\mathbf{x}, z, t)$ into (3.13).

5. Mean products of three first-order quantities

A number of the higher order quantities, such as the difference (3.22) between the potential and kinetic energy densities, presented themselves as the average product of three first-order terms. One would expect, in view of the observed closeness to which the surface displacement distribution approaches the Gaussian form, that third-order correlations (i.e. dimensionless covariances) would be numerically small. However, we can show that they are small in an order of magnitude sense, or, more precisely, that in a homogeneous stationary wave field *any mean triple product of first-order terms is of fourth (or higher) order.*

For the proof of this statement, we observe first that any first-order quantity can be represented (correct to this order) as a generalized Fourier integral involving $A(\mathbf{k})$ multiplied by algebraic factors whose nature depends on the particular quantity concerned. The expression (4.4) for $\phi(\mathbf{x}, z, t)$ is an illustration of this, and clearly $\phi_\xi(\mathbf{x}, t)$, $\mathbf{u}_\xi(\mathbf{x}, t)$, etc., are of the same form.

Let $f_1(\mathbf{x}, t)$, $f_2(\mathbf{x}+\mathbf{r}, t+\tau_2)$ and $f_3(\mathbf{x}+\mathbf{s}, t+\tau_3)$ represent three first-order quantities measured at positions $\mathbf{x}$, $\mathbf{x}+\mathbf{r}$, $\mathbf{x}+\mathbf{s}$ and times t, $t+\tau_2$, $t+\tau_3$ respectively. Then the mean triple product can be represented, correct to the third order, as

$$\overline{f_1 f_2 f_3} = \int_{\mathbf{k}_1}\int_{\mathbf{k}_2}\int_{\mathbf{k}_3} K(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\,\overline{A^*(\mathbf{k}_1)A(\mathbf{k}_2)A(\mathbf{k}_3)}$$
$$\times \exp -i\{(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3).\mathbf{x}-(\omega_1-\omega_2-\omega_3)t\}$$
$$\times \exp i\{\mathbf{k}_2.\mathbf{r}-\omega_2\tau_2\}\exp i\{\mathbf{k}_3.\mathbf{s}-\omega_3\tau_3\}\,d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

where K is an algebraic function and $\omega_i = (gk_i)^{\frac{1}{2}}$ for $i = 1, 2, 3$. The simplest argument is as follows.† If the left-hand side of this expression is to be independent of position $\mathbf{x}$ and time origin t, then the only non-zero contributions to the integral can occur only when both

$$\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3 = 0, \quad \omega_1-\omega_2-\omega_3 = 0. \tag{5.1}$$

From the second of these conditions,

$$k_1^{\frac{1}{2}} = k_2^{\frac{1}{2}}+k_3^{\frac{1}{2}},$$

or

$$\frac{k_1}{k_2+k_3} = 1+\frac{2(k_2k_3)^{\frac{1}{2}}}{k_2+k_3} > 1. \tag{5.2}$$

But from the first of the conditions (5.1), $-\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ form a closed triangle, so that $k_1/(k_2+k_3) < 1$, which is incompatible with (5.2) except for the degenerate case when either k_2 or k_3 vanishes. This contingency can be excluded since it would imply the existence of covariances of the kind above that were independent of $\mathbf{r}$ or $\mathbf{s}$. The conclusion is that there are no non-zero contributions to the integral, and that any triple covariance of this kind is zero to the third order.

Two comments on this rather general result might be pertinent. The result depends crucially on the dispersive nature of the wave, where a wave-number $\mathbf{k}$ is associated with a unique frequency which depends on but is not proportional to $k = |\mathbf{k}|$. In a non-dispersive wave system where ω is proportional to k, the

† An alternative but slightly longer procedure involves inversion of this expression.

conditions (5.1) are satisfied by triads of parallel vectors. Also if the frequency is not a unique function of scalar wave-number, as in the *second* approximation to a gravity-wave field, we do not have the simple separation (3.2) and the result fails. Secondly, it might be noted that the reasons for this result are precisely the same as those for the absence of any continuing energy transfer from one wave-number to another in second-order gravity wave theory (Phillips 1960*a*). The conditions required for such a transfer are equivalent to (5.1) which are incapable of satisfaction.

An immediate consequence of this result is that the difference between the potential and kinetic energy density is of fourth order. Since T and V are themselves second-order quantities, the fractional difference is of order $\overline{(\nabla\xi)^2}$ since $[\overline{(\nabla\xi)^2}]^{\frac{1}{2}}$ is our representative first-order quantity. The energy partition is thus very nearly equal; even in a well-developed sea where $[\overline{(\nabla\xi)^2}]^{\frac{1}{2}}$ is of order 0·15 the energy densities are equal to within a few per cent.

A second consequence is an interesting one that can be compared directly with observation, though it is not involved in the discussion of § 3. According to the above result, $\overline{\xi^3}$ is of fourth order or smaller, whereas $\overline{\xi^2}$ is clearly of second order. It follows that the skewness of the probability distribution of the surface displacement, namely

$$\overline{\xi^3}/[\overline{\xi^2}]^{\frac{3}{2}}, \tag{5.3}$$

is a first-order dimensionless quantity, and so is of order $[\overline{(\nabla\xi)^2}]^{\frac{1}{2}}$. In other words,

$$\frac{\overline{\xi^3}}{[\overline{(\nabla\xi)^2}]^{\frac{1}{2}}\,[\overline{\xi^2}]^{\frac{3}{2}}} \tag{5.4}$$

is of order unity or less. As $[\overline{(\nabla\xi)^2}]^{\frac{1}{2}} \to 0$, the skewness (5.3) tends to zero also, in accord with the well known property that in the limit of infinitesimal wave slopes, the surface displacement has a symmetric probability distribution. As the root mean square slope increases, so does the skewness, reflecting the tendency of the waves to form sharper crests and shallower troughs. A direct calculation, using the second-order terms, confirms that the quantity (5.4) is of order unity, although the resulting expression, involving ratios of integrals involving the spectrum $\Phi(\mathbf{k})$ is too complicated to be of much value in a general case. A second direct calculation for a single train of finite amplitude waves shows that the quantity (5.4) is 3/8 plus a term of order (slope)^2, in agreement with our order-of-magnitude result.

The prediction that the quantity (5.4) is quite generally of order unity can be compared with some observations made by Kinsman (1960) on wind-generated waves in a branch of the Chesapeake Bay. He measured properties of the waves set up by a steady wind blowing over a fixed fetch. The conditions were chosen so that the wave field was stationary in time and the rate of growth with increasing fetch was slow, and so they approximated to the conditions envisaged in the theorem above. Kinsman measured the skewness for each of 24 sets of observations, and the root mean square slope was calculated approximately from his measured spectra. Table 1 gives the results of these calculations, and it is evident that the quantity (5.4) is indeed of order unity or less. The sets numbered between 81 and 88 gave exceptionally low (even negative) values for the skewness, but in no case is the ratio (5·4) of order larger than unity, and the variation from one observation to another is no more than is to be expected from the widely differing properties of the wave field generated on the different occasions.

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Observation number	Skewness	$\dfrac{\text{Skewness}}{[\overline{(\nabla\xi)^2}]^{\frac{1}{2}}}$
9	0·172	2·3
10	0·143	2·1
11	0·096	1·4
12	0·182	2·4
17	0·175	2·1
18	0·219	2·4
27	0·158	2·1
28	0·178	2·3
67	0·082	1·1
68	0·087	1·1
69	0·067	1·0
70	0·027	0·4
75	0·101	1·5
76	0·092	1·4
81	0·029	0·4
82	0·034	0·5
83	−0·002	0
84	0·044	0·7
85	−0·005	−0·1
86	0·011	0·2
87	0·005	0·1
88	−0·046	−0·7
93	0·144	2·1
94	0·136	2·1

TABLE 1

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